3.2 Construction of the heat kernels

3.2.1 Toy model

In this section, we will construct the heat kernel for generalized Laplacian H on $V := \mathscr{C}^\infty(M, E)$.

We study the toy model first: let V be a finite dimensional vector space and H be a linear endomorphism; we construct $P_t = e^{-tH}$.

Definition 3.2.1. The k-simplex

$$\Delta_k := \{(t_1, \cdots, t_k) : 0 \leq t_1 \leq \cdots \leq t_k \leq 1\} \subset \mathbb{R}^k. \tag{3.2.1}$$

We often parametrize Δ_k by the coordinates

$$\sigma_0 = t_1, \sigma_i = t_{i+1} - t_i, \; 1 \leq i \leq k-1, \; \sigma_k = 1 = t_k, \tag{3.2.2}$$

such that $\sigma_1 + \cdots + \sigma_k = 1$ and $0 \leq \sigma_i \leq 1$. For $t > 0$, the rescaled simplex

$$t\Delta_k := \{(t_1, \cdots, t_k) : 0 \leq t_1 \leq \cdots \leq t_k \leq t\} \subset \mathbb{R}^k. \tag{3.2.3}$$

Let v_k be the volume of Δ_k. since $v_1 = 1$ and

$$v_k = \int_0^1 \mathrm{vol}(t_k \Delta_{k-1}) dt_k = \int_0^1 t_k^{k-1} v_{k-1} dt_k = \frac{v_{k-1}}{k}, \tag{3.2.4}$$

we have

$$v_k = \frac{1}{k!}. \tag{3.2.5}$$

Let $K_t : \mathbb{R}_+ \to \mathrm{End}(V)$ be an approximate solution of the heat equation for small t in the sense that for some small $\alpha > 0$, there exists $C > 0$ such that

$$R_t = \frac{dK_t}{dt} + HK_t \leq Ct^\alpha, \tag{3.2.6}$$

and $K_0 = 1$. The function K_t is also called a parametrix for the heat equation. The function R_t is called the remainder.

Let

$$Q_t^1 = \int_0^t K_{t-t_1} R_{t_1} dt_1. \tag{3.2.7}$$

Then

$$\frac{dQ_t^1}{dt} = R_t + \int_0^t R_{t-t_1} R_t dt_1 - HQ_t^1. \tag{3.2.8}$$

So from (3.2.6)-(3.2.8),

$$\left(\frac{d}{dt} + H\right)(K_t - Q_t^1) = -\int_0^t R_{t-t_1} R_t dt_1 = O(t^{2\alpha+1}). \tag{3.2.9}$$

Following this way, we could make the error term smaller and smaller:

Theorem 3.2.2. *Let $Q_t : \mathbb{R}_+ \to \mathrm{End}(V)$ be defined by*

$$Q_t^k := \int_{t\Delta_k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_k \tag{3.2.10}$$

and $Q_t^0 = K_t$. Then

$$P_t = e^{-tH} = \sum_{k=0}^{+\infty} (-1)^k Q_t^k \tag{3.2.11}$$

and

$$P_t = K_t + O(t^{1+\alpha}). \tag{3.2.12}$$

Proof. Let

$$R^{(k)}(s) := \int_{s\Delta_{k-1}} R_{s-t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1} dt_1 \cdots dt_k. \tag{3.2.13}$$

Then as in (3.2.8), we have

$$\left(\frac{d}{dt} + H\right) Q_t^k = R^{(k+1)}(t) + R^{(k)}(t). \tag{3.2.14}$$

From (3.2.6) and (3.2.3),

$$R^{(k)}(t) = O(t^{k\alpha}). \tag{3.2.15}$$

Since $\mathrm{vol}(t\Delta_k) = t^k/k!$, there exists $C_0 > 0$ such that

$$|Q_t^k| \le C_0 C^k t^{k\alpha} \frac{t^k}{k!}. \tag{3.2.16}$$

So the right hand side of (3.2.11) converges. Since $Q_0^0 = 1$ and $Q_0^k = 0$ for $k > 0$, we obtain (3.2.11). From (3.2.16) again, we have $P_t = K_t + O(t^{1+\alpha})$.

The proof of Theorem 3.2.2 is completed. □

The original version of Theorem 3.2.2 is the Volterra series for the exponential of a perturbed operator. If $H = H_0 + H_1 \in \mathrm{End}(V)$, for $K_t = e^{-tH_0}$,

$$R_t = \left(\frac{d}{dt} + H\right) e^{-tH_0} = H_1 e^{-tH_0}. \tag{3.2.17}$$

So from Theorem 3.2.2, we have

$$e^{-t(H_0+H_1)} = e^{-tH_0} + \sum_{k=1}^{\infty} (-1)^k I_k, \tag{3.2.18}$$

where

$$\begin{aligned} I_k :&= \int_{t\Delta_k} e^{-(t-t_k)H_0} H_1 e^{-(t_k - t_{k-1})H_0} \cdots H_1 e^{-t_1 H_0} dt_1 \cdots dt_k \\ &= \int_{t\Delta_k} e^{-\sigma_0 H_0} H_1 e^{-\sigma_1 H_0} \cdots H_1 e^{-\sigma_k H_0} d\sigma_1 \cdots d\sigma_k. \end{aligned} \tag{3.2.19}$$

So

$$\begin{aligned} e^{-t(H_0+H_1)} &= \sum_{k=0}^{\infty} (-t)^k e^{-\sigma_0 t H_0} H_1 e^{-\sigma_1 t H_0} \cdots H_1 e^{-\sigma_k t H_0} d\sigma_1 \cdots d\sigma_k \\ &= e^{-tH_0} - t \int_0^1 e^{(1-\sigma)tH_0} H_1 e^{-\sigma t H_0} d\sigma + \cdots \end{aligned} \tag{3.2.20}$$

3.2.2 Estimates of the parametrix

For $V = \mathscr{C}^\infty(M, E)$, we study the kernel instead of the operator.

We leave the proof of the following theorem to the next subsection.

Theorem 3.2.3. *For every $N \in \mathbb{Z}_+$, there exists a smooth one-parameter family of smooth kernels $k_t^N(x, y)$, such that for every $\ell \in \mathbb{N}$,*

(1) for every $T > 0$, there exists $C > 0$ such that for $0 < t < T$, $u \in \mathscr{C}^\infty(M, E)$, we have

$$\|K_t^N u\|_{\mathscr{C}^\ell} \le C \|u\|_{\mathscr{C}^\ell}, \tag{3.2.21}$$

where K_t^N is the operator associated with $k_t^N(x, y)$;

(2) for $u \in \mathscr{C}^\infty(M, E)$,

$$\lim_{t \to 0} \|K_t u - u\|_{\mathscr{C}^\ell} = 0; \tag{3.2.22}$$

(3) there exists $C(\ell, s) > 0$ such that the kernel

$$r_t^N(x,y) := (\partial_t + H_x)k_t^N(x,y) \tag{3.2.23}$$

satisfies the estimate

$$\|\partial_t^s r_t^N\|_{\mathscr{C}^\ell} \leq C(\ell,s)t^{N-n/2-\ell/2-s}. \tag{3.2.24}$$

In order to simplify the notation, we omit the symbols "N" and "$dt_1 \cdots dt_k$" if there is no confuse.

Let K_t and R_t be the corresponding operators with respect to k_t and r_t.

As in (3.2.10) and (3.2.13), we consider

$$Q_t^k := \int_{t\Delta_k} K_{t-t_k} R_{t_k - t_{k-1}} \cdots R_{t_2 - t_1} R_{t_1}, \tag{3.2.25}$$

which is defined by the kernel

$$q_t^k(x,y) = \int_{t\Delta_k}\int_{M^k} k_{t-t_k}(x,z_k) r_{t_k-t_{k-1}}(z_k,z_{k-1}) \cdots r_{t_1}(z_1,y) \tag{3.2.26}$$

and the kernel

$$r_t^{k+1}(x,y) = \int_{t\Delta_k}\int_{M^k} r_{t-t_k}(x,z_k) r_{t_k-t_{k-1}}(z_k,z_{k-1}) \cdots r_{t_1}(z_1,y). \tag{3.2.27}$$

Lemma 3.2.4. *If $N > (n+\ell)/2$, then for $s \in \mathbb{N}$,*

$$\|\partial_t^s r_t^{k+1}\|_{\mathscr{C}^\ell(M\times M)} \leq C^{k+1} t^{(k+1)(N-n/2)-\ell/2-s}\,\mathrm{vol}(M)^k \frac{t^k}{k!}. \tag{3.2.28}$$

Proof. If $N > (n+\ell)/2$, by (3.2.24), r_t and its derivatives up to order ℓ extend continuously to $t = 0$. Using (3.2.24) again, we obtain Lemma 3.2.4.

The proof of Lemma 3.2.4 is completed. □

Lemma 3.2.5. *Assume that $N > (n+\ell)/2$ and that $\ell \geq 1$.*

(1) There exists $\widetilde{C} > 0$ such that for every $k \geq 1$,

$$\|q_t^k\|_{\mathscr{C}^\ell(M\times M)} \leq \widetilde{C}C^k\,\mathrm{vol}(M)^{k-1} t^{k(N-n/2)-\ell/2} \frac{t^k}{(k-1)!}. \tag{3.2.29}$$

(2) The kernel $q_t^k(x,y)$ is $\mathscr{C}^1$ on t and

$$(\partial_t + H_x)q_t^k(x,y) = r_t^{k+1}(x,y) + r_t^k(x,y). \tag{3.2.30}$$

Proof. Let

$$b(t, s, x, y) = \int_{z \in M} k_{t-s}(x, z) r_s^k(z, y) = K_{t-s,x} r_s^k(x, y). \tag{3.2.31}$$

Then from (3.2.26) and (3.2.27),

$$q_t^k(x, y) = \int_0^t b(t, s, x, y) ds. \tag{3.2.32}$$

From Theorem 3.2.3 (1), Lemma 3.2.4 and (3.2.31), for $0 \le s \le t$,

$$\|b(t, s)\|_{\mathscr{C}^\ell(M \times M)} \le C' C^k \operatorname{vol}(M)^{k-1} t^{k(N-n/2)-\ell/2} \frac{t^{k-1}}{(k-1)!}. \tag{3.2.33}$$

So we obtain (1) from (3.2.32) and (3.2.33).

From (3.2.23) and (3.2.31), $b(t, s, x, y)$ is continuous on s, $\mathscr{C}^1$ on t and smooth on x. From (3.2.24),

$$(\partial_t + H_x) b(t, s, x, y) = \int_{z \in M} r_{t-s}(x, z) r_s^k(z, y) = r^{k+1}(x, y). \tag{3.2.34}$$

Then (3.2.30) follows from (3.2.32) and (3.2.34).

The proof of Lemma 3.2.5 is completed. □

Theorem 3.2.6. *Assume that the kernel $k_t^N(x, y)$ satisfies the conditions of Theorem 3.2.3 with $N > n/2 + 1$.*

(1) For any ℓ such that $N > (n + \ell + 1)/2$,

$$p_t(x, y) = \sum_{k=0}^{\infty} (-1)^k q_t^k(x, y) \tag{3.2.35}$$

converges in the $\mathscr{C}^{\ell+1}(M \times M)$-norm and defines a $\mathscr{C}^1$-map from $\mathbb{R}_+$ to $\mathscr{C}^\ell(M \times M, E \boxtimes E^*)$ such that

$$(\partial_t + H_x) p_t(x, y) = 0. \tag{3.2.36}$$

(2) When $t \to 0$,

$$\|\partial_t^s (p_t - k_t^N)\|_{\mathscr{C}^\ell(M \times M)} = O(t^{(N-n/2)-s-\ell/2+1}). \tag{3.2.37}$$

(3) The kernel p_t is a heat kernel for the operator H.

Proof. From (3.2.29),

$$\sum_{k=0}^{\infty} \|q_t^k\|_{\mathscr{C}^{\ell+1}(M\times M)} \leq \|k_t\|_{\mathscr{C}^{\ell+1}(M\times M)} + \widetilde{C} C t^{N-n/2-\ell/2+1/2} e^{C\,\mathrm{vol}(M) t^{N-n/2+1}} < +\infty. \quad (3.2.38)$$

So (3.2.25) converges in $\mathscr{C}^{\ell+1}(M\times M)$-norm. From (3.2.28), (3.2.29) and (3.2.30),

$$\begin{aligned}\|\partial_t q_t^k\|_{\mathscr{C}^{\ell}(M\times M)} &\leq \|r_t^{k+1}\|_{\mathscr{C}^{\ell}(M\times M)} + \|r_t^k\|_{\mathscr{C}^{\ell}(M\times M)} + \|q_t^k\|_{\mathscr{C}^{\ell+2}(M\times M)} \\ &\leq C^{k+1} t^{(k+1)(N-n/2)-\ell/2}\,\mathrm{vol}(M)^k \frac{t^k}{k!} + C^k t^{k(N-n/2)-\ell/2}\,\mathrm{vol}(M)^{k-1} \frac{t^{k-1}}{(k-1)!} \\ &\quad + \widetilde{C} C^k\,\mathrm{vol}(M)^{k-1} t^{k(N-n/2)-\ell/2-1} \frac{t^k}{(k-1)!} \\ &\leq C t^{N-n/2-\ell/2} (C\,\mathrm{vol}(M) t^{N-n/2+1}/k + 1 + \widetilde{C}) \\ &\quad \cdot C^{k-1} t^{(k-1)(N-n/2)}\,\mathrm{vol}(M)^{k-1} \frac{t^{k-1}}{(k-1)!}. \end{aligned} \quad (3.2.39)$$

So there exist $C_0, C_1 > 0$ such that

$$\sum_{k=0}^{\infty} \|\partial_t q_t^k\|_{\mathscr{C}^{\ell}(M\times M)} \leq \|\partial_t k_t\|_{\mathscr{C}^{\ell}(M\times M)} + (C_0 t^{N-n/2+1} + C_1) t^{N-n/2-\ell/2} e^{C\,\mathrm{vol}(M) t^{N-n/2+1}} < +\infty. \quad (3.2.40)$$

Thus p_t is $\mathscr{C}^1$ on t from $\mathbb{R}_+$ to $\mathscr{C}^\ell(M\times M)$. As in Theorem 3.2.2, we have (3.2.26).

From the proof of (3.2.38) and (3.2.40), we get (3.2.37).

For (3), we only need to check the initial condition. Since k_t^N satisfies the initial condition, from (3.2.37), we get (3).

The proof of Theorem 3.2.6 is completed. □

3.2.3 Formal solution

In this subsection, we will prove Theorem 3.2.3. For the complexity of the construction, we will start from some basic results in Riemannian Geometry.

Let g^{TM} be the metric on M. Usually we denote it by g for simplicity. we consider a smooth path $x_t : [0,1] \to M$ and define its length as

$$L(x_t) = \int_0^1 |\dot{x}_t| dt. \quad (3.2.41)$$

The Riemannian distance between $x_0, x_1 \in M$ is the infimum of $L(x_t)$ over all smooth paths connecting them, denoted by $d(x, y)$. Let ∇ be the Levi-Civita connection. A smooth path is a geodesic if for any $t \in [0,1]$,

$$\nabla_{\dot{x}_t}\dot{x}_t = 0. \tag{3.2.42}$$

Given $\mathbf{x} = \dot{x}_0 \in T_{x_0}M$ small enough, the solution of (3.2.42) is unique. We write $x_1 = \exp_{x_0}\mathbf{x}$. Since the derivative $\exp_{x_0,*}$ is an isomorphism, by the inverse function theorem, $\exp_{x_0}$ defines a diffeomorphism from a small ball around zero to a neighborhood of x_0 in M. Let inj_{x_0} be the radius of the largest ball such that $\exp_{x_0}$ is a diffeomorphism. Let $\mathrm{inj} = \inf_{x\in M}\mathrm{inj}_{\mathrm{x}}$. Since M is compact, $\mathrm{inj} > 0$. Choose an orthonormal frame of $T_{x_0}M$, on which the coordinate functions are $\mathbf{x}_i$ and the partial derivatives are $\frac{\partial}{\partial \mathbf{x}_i}$. On $B(0,\varepsilon) \subset T_{x_0}M$, we define the metric by $\exp^*_{x_0}(g)$. If we consider $B(0,\varepsilon)$ as a chart of M at x_0, we have $\frac{\partial}{\partial \mathbf{x}_i} = \frac{\partial}{\partial x_i}$. Usually, we simply denoted it by ∂_i. With respect to this coordinates, we have

$$(\partial_i, \partial_j) = g_{ij}. \tag{3.2.43}$$

Let

$$\mathcal{R} = \sum_i \mathbf{x}_i \partial_i \in T_{\mathbf{x}}(T_{x_0}M). \tag{3.2.44}$$

Then $\exp_{x_0,*}\mathcal{R} \in T_{\exp_{x_0}\mathbf{x}}M$, which we also denote by $\mathcal{R}$.

In order to distinguish the points on $T_{x_0}M$ and those on M, we write $x = \exp_{x_0}\mathbf{x}$. Let x_t be the geodesic connecting x_0 and x, and let $Y(t) \in T_{x_t}M$ be a vector field along x_t. if for any $t \in [0,1]$,

$$\nabla_{\dot{x}_t} Y(t) = 0, \tag{3.2.45}$$

we say $Y(1)$ **is the parallel transport of** $Y(0)$ **along** x_t. Since $\mathbf{x} \in B(0, \mathrm{inj})$, the solution of (3.2.45) is unique associated with initial condition. So $Y(1)$ is uniquely determined by $Y(0)$. We write

$$Y(1) = \tau(x, x_0)Y(0). \tag{3.2.46}$$

Let

$$e_i(x) := \tau(x, x_0)\partial_i. \tag{3.2.47}$$

Lemma 3.2.7. *(1)* $\{e_i(x)\}$ *is an orthonormal frame of* T_xM.

(2) $e_i(x) = \partial_i + O(|\mathbf{x}|)$.

(3) $\nabla_{\mathcal{R}} e_i = 0$.

Proof. (1) Let $Y_0(t), Y_1(t) \in T_{x_t}M$ be vector fields along x_t satisfying (3.2.45). Since ∇^{TM} preserves the metric,

$$\dot{x}_t(Y_0(t), Y_1(t)) = (\nabla_{\dot{x}_t} Y_0(t), Y_1(t)) + (Y_0(t), \nabla_{\dot{x}_t} Y_1(t)) = 0. \tag{3.2.48}$$

So $(Y_0(t), Y_1(t))$ is a constant along x_t. Let $Y_0(0) = \partial_i$ and $Y_1(0) = \partial_j$, we get (1).

(2) Let $e_i(x) = f_{ij}(x)\partial_j$. Note that $f_{ij}(0) = \delta_{ij}$. So $f_{ij}(x) = \delta_{ij} + O(|\mathbf{x}|)$. We get (2).

(3) Let $x_t = \exp_{x_0} t\mathbf{x}$. Then $\mathcal{R} = |\mathcal{R}|\dot{x}_t$. Since $\nabla_{\dot{x}_1} e_i = 0$, we get (3).

The proof of Lemma 3.2.7 is completed. □

Lemma 3.2.8. *(1)* $\nabla_{\mathcal{R}}\mathcal{R} = 0$.

(2) $\mathcal{R} = \sum_i \mathbf{x}_i e_i$, *and thus* $(\mathcal{R}, \mathcal{R}) = |\mathbf{x}|^2$.

(3) $(\mathcal{R}, \partial_i) = \mathbf{x}_i$, *and thus* $\mathbf{x}_i = g_{ij}\mathbf{x}_j$.

(4) $d(x_0, x) = |\mathbf{x}|$. *(Note that* $|\mathbf{x}|$ *only depends on* $g(x_0)$ *but* $d(x_0, x)$ *depends on* $g(x_t)$ *for* $t \in [0, 1]$*.)*

Proof. (1) The curve $\mathbf{x}_t = t\mathbf{x}$ is a geodesic. Note that $\mathcal{R}(\mathbf{x}_t) = t\dot{\mathbf{x}}_t$. We also simply denote by $\nabla = \exp_{x_0}^*(\nabla)$. So by (3.2.42),

$$\nabla_{\mathcal{R}}\mathcal{R} = t\nabla_{\dot{\mathbf{x}}_t}(t\dot{\mathbf{x}}_t) = t\dot{\mathbf{x}}_t = \mathcal{R}. \tag{3.2.49}$$

For the second equality, we consider function $f(x_t) = t$ and then $\nabla_{\dot{\mathbf{x}}_t} t = \nabla_{\dot{\mathbf{x}}_t} f = \frac{\partial}{\partial t} f(x_t) = 1$.

(2) From Lemma 3.2.7 (3) and (1),

$$\mathcal{R}(\mathcal{R}, e_i) = (\nabla_{\mathcal{R}}\mathcal{R}, e_i) + (\mathcal{R}, \nabla_{\mathcal{R}} e_i) = (\mathcal{R}, e_i). \tag{3.2.50}$$

From Lemma 3.2.7 (2),

$$(\mathcal{R}, e_i) = \sum_j \mathbf{x}_j(\partial_j, e_i) = \mathbf{x}_i + O(|\mathbf{x}|^2). \tag{3.2.51}$$

Since $\mathcal{R}(\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}) = k\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}$, from (3.2.50), there is no $O(|\mathbf{x}|^2)$ term in (3.2.51). So $(\mathcal{R}, e_i) = \mathbf{x}_i$. Since $\{e_i(x)\}$ is an orthonormal frame by Lemma 3.2.7 (1), $\mathcal{R} = \sum_i \mathbf{x}_i e_i$ and $(\mathcal{R}, \mathcal{R}) = |\mathbf{x}|^2$.

(3) Note that

$$[\mathcal{R}, \partial_i] = -\partial_i(\mathbf{x}_j)\partial_j = -\partial_i. \tag{3.2.52}$$

Since ∇ is torsion free,

$$(\mathcal{R}, \nabla_{\mathcal{R}}\partial_i) = (\mathcal{R}, \nabla_{\partial_i}\mathcal{R}) + (\mathcal{R}, [\mathcal{R}, \partial_i]) = \frac{1}{2}\partial_i|\mathcal{R}|^2 - (\mathcal{R}, \partial_i). \tag{3.2.53}$$

So from (2) and (3.2.53),

$$\mathcal{R}(\mathcal{R}, \partial_i) = (\nabla_{\mathcal{R}}\mathcal{R}, \partial_i) + (\mathcal{R}, \nabla_{\mathcal{R}}\partial_i) = \frac{1}{2}\partial_i|\mathcal{R}|^2 = \mathbf{x}_i. \tag{3.2.54}$$

Since $(\mathcal{R}, \partial_i) = \sum_j \mathbf{x}_j(\partial_i, \partial_j) = \mathbf{x}_i + O(|\mathbf{x}|^2)$, by $\mathcal{R}(\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}) = k\mathbf{x}_{i_1} \cdots \mathbf{x}_{i_k}$ again, from (3.2.54), we have $(\mathcal{R}, \partial_i) = \mathbf{x}_i$, and thus $\mathbf{x}_i = g_{ij}\mathbf{x}_j$.

(4) We use the fact that locally the geodesic is the shortest path. Let $\mathbf{x}_t = t\mathbf{x}$. From (2),

$$\begin{aligned} d(x_0, x) = \int_0^1 |\dot{\mathbf{x}}_t| dt = \int_0^1 t^{-1}|R(\dot{\mathbf{x}}_t)| dt \\ = \int_0^1 t^{-1}|\mathbf{x}_t| dt = \int_0^1 |\mathbf{x}| dt = |\mathbf{x}|. \end{aligned} \tag{3.2.55}$$

The proof of Lemma 3.2.8 is completed. □

Let

$$j(\mathbf{x}) = \det{}^{1/2}(g_{ij}(\mathbf{x})). \tag{3.2.56}$$

Then the pull back of the volume form on $T_{x_0}M$

$$dx = j(\mathbf{x})d\mathbf{x}. \tag{3.2.57}$$

In other words, we have

$$j(\mathbf{x}) = |\det(d_{\mathbf{x}} \exp_{x_0})|. \tag{3.2.58}$$

Take $\varepsilon <$ inj. Let $V_y = \mathrm{Im}(\exp_y |_{B(0,\varepsilon)})$. For $x \in V_y$, we define a neighborhood of the diagonal of $M \times M$ by

$$U_\varepsilon = \{(x, y) \in M \times M : x \in V_y\}. \tag{3.2.59}$$

If $(x, y) \in U_\varepsilon$, $d(x, y) < \varepsilon$.

As in (3.1.30), let

$$q_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} \in \mathscr{C}^\infty(\mathbb{R}_+ \times U_\varepsilon), \tag{3.2.60}$$

which is modelled on the Euclidean heat kernel. To construct an approximate solution to the heat equation for H, we plan to find a formal solution to the heat equation as a series of the form

$$k_t(x, y) = q_t(x, y) \sum_{t=0}^{\infty} t^i \Psi_i(x, y, H), \tag{3.2.61}$$

where the coefficients $(x, y) \mapsto \Psi_i(x, y, H)$ are smooth sections of the bundle $E \boxtimes E^*$ over U_ε.

We would like to have $(\partial_t + H)k_t(x, y) = 0$.

Proposition 3.2.9. *For any time dependent section s_t of E over U_ε, we have*

$$(\partial_t + H)(q_t \cdot s_t) = q_t \cdot \big(\partial_t + H + t^{-1}\nabla^E_{\mathcal{R}} + (2t)^{-1}\mathcal{R}(\log j)\big)s_t. \tag{3.2.62}$$

Proof. From (1.4.35),

$$\begin{aligned}\Delta^E(q_t s_t) &= -\nabla^E_{e_i}\nabla^E_{e_i}(q_t s_t) + \nabla^E_{\nabla_{e_i}e_i}(q_t s_t)\\ &= -\nabla^E_{e_i}(e_i(q_t)s_t - q_t\nabla^E_{e_i}s_t) + (\nabla_{e_i}e_i)(q_t)s_t + q_t\nabla^E_{\nabla_{e_i}e_i}s_t\\ &= (\Delta^E q_t)s_t - 2e_i(q_t)\nabla^E_{e_i}s_t + q_t\Delta^E s_t.\end{aligned} \tag{3.2.63}$$

From (3.2.63),

$$(\partial_t + H)(q_t s_t) = ((\partial_t + H)q_t)s_t - 2(dq_t, \nabla^E s_t) + q_t((\partial_t + H)s_t). \tag{3.2.64}$$

Write $x = \exp_y \mathbf{x}$. Then from Lemma 3.2.8 (4),

$$q_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}. \tag{3.2.65}$$

Then

$$\partial_t q_t = \left(-\frac{n}{2t} + \frac{|\mathbf{x}|^2}{4t^2}\right) q_t. \tag{3.2.66}$$

From (3.2.65),

$$\begin{aligned}\Delta q_t &= -\frac{1}{(4\pi t)^{n/2}} e_i\left(-\frac{1}{4t}e^{-\frac{|\mathbf{x}|^2}{4t}} e_i(|\mathbf{x}|^2)\right) - \frac{1}{(4\pi t)^{n/2}}\frac{1}{4t}e^{-\frac{|\mathbf{x}|^2}{4t}}(\nabla_{e_i}e_i)(|\mathbf{x}|^2)\\ &= -\frac{1}{4t}q_t \cdot \left(\Delta(|\mathbf{x}|^2) + \frac{1}{4t}\big(e_i(|\mathbf{x}|^2)\big)^2\right).\end{aligned} \tag{3.2.67}$$

We calculate $\Delta(|\mathbf{x}|^2)$ first. Let $g^{ij} = (g^{-1})_{ij}$. Then $g^{ij}g_{jk} = \delta_{ik}$. From Lemma 3.2.8 (3),

$$g^{ij}\mathbf{x}_j = g^{ij}g_{jk}\mathbf{x}_k = \mathbf{x}_i. \tag{3.2.68}$$

For any $\phi \in \mathscr{C}^\infty_0(T_yM)$, from (1.4.37), (3.2.57) and (3.2.68),

$$\begin{aligned}\int_{T_yM}\phi\Delta(|\mathbf{x}|^2)dx &= \int_{T_yM}(d\phi, d(|\mathbf{x}|^2))dx = 2\int_{T_yM}(\partial_i\phi)g^{ij}\mathbf{x}_j j(\mathbf{x})d\mathbf{x}\\ &= 2\int_{T_yM}(\partial_i\phi)\mathbf{x}_i j(\mathbf{x})d\mathbf{x} = -2\int_{T_yM}\phi\partial_i(\mathbf{x}_i j(\mathbf{x}))d\mathbf{x}\\ &= -2\int_{T_yM}\phi(n + \mathcal{R}(\log j))dx.\end{aligned} \tag{3.2.69}$$

So

$$\Delta(|\mathbf{x}|^2) = -2(n + \mathcal{R}(\log j)). \tag{3.2.70}$$

On the other hand, from (3.2.68),

$$\big(e_i(\|\mathbf{x}\|^2)\big)^2 = g^{ij}\partial_i(\|\mathbf{x}\|^2)\partial_j(|\mathbf{x}|^2) = 4g^{ij}x_i x_j = 4x_j x_j = 4|\mathbf{x}|^2. \tag{3.2.71}$$

By (3.2.67), (3.2.70) and (3.2.71), we have

$$\Delta q_t = \left(\frac{1}{2t}(n + \mathcal{R}(\log j)) - \frac{|\mathbf{x}|^2}{4t^2}\right) q_t. \tag{3.2.72}$$

From (3.2.65),

$$\partial_i(q_t) = -\frac{\partial_i(|\mathbf{x}|^2)}{4t} q_t = -\frac{x_i}{2t} q_t. \tag{3.2.73}$$

So from (3.2.69),

$$(dq_t, \nabla^E s_t) = g^{ij}\partial_i(q_t)\nabla^E_{\partial_j} s_t = -\frac{1}{2t} q_t \nabla^E_{\mathcal{R}} s_t. \tag{3.2.74}$$

Therefore, (3.2.62) is obtained from (3.2.66), (3.2.72) and (3.2.74).

The proof of Proposition 3.2.9 is completed. □

Let

$$B = j^{1/2} \circ H \circ j^{-1/2}. \tag{3.2.75}$$

Let

$$\Phi_t = j^{1/2} s_t. \tag{3.2.76}$$

Proposition 3.2.10. *The following identity holds,*

$$\big(\partial_t + H + t^{-1}\nabla^E_{\mathcal{R}} + (2t)^{-1}\mathcal{R}(\log j)\big)s_t = j^{-1/2} \cdot \big(\partial_t + B + t^{-1}\nabla^E_{\mathcal{R}}\big)\Phi_t. \tag{3.2.77}$$

Proof. From (3.2.75),

$$H \circ j^{-1/2} = j^{-1/2} B. \tag{3.2.78}$$

From (3.2.76),

$$j^{-1/2}\nabla^E_{\mathcal{R}}\Phi_t = j^{-1/2}\nabla_{\mathcal{R}} j^{1/2} s_t = \frac{1}{2} j^{-1}\mathcal{R}(j) s_t + \nabla^E_{\mathcal{R}} s_t. \tag{3.2.79}$$

So (3.2.77) follows from (3.2.78) and (3.2.79).

The proof of Proposition 3.2.10 is completed. □

Definition 3.2.11. Let $\Phi_t(x,y)$ be a formal power series in t whose coefficients are smooth sections of $E \boxtimes E^*$ on U_ε. We say $q_t(x,y)j^{-1/2}(\mathbf{x})\Phi_t(x,y)$ is a **formal solution** of the heat equation around y if $x \mapsto \Phi_t(x,y)$, considered as a section of the bundle $E \boxtimes E^*$ over V_y, satisfies the equation

$$\left(\partial_t + B + t^{-1}\nabla^E_{\mathcal{R}}\right)\Phi_t(\cdot, y) = 0. \tag{3.2.80}$$

Let $x_t = \exp_y t\mathbf{x}$ be the geodesic connecting y and x, and let $Y(t) \in E_{x_t}$. As in (3.2.45), if for any $t \in [0,1]$,

$$\nabla^E_{\dot{x}_t} Y(t) = 0, \tag{3.2.81}$$

we say $Y(1)$ **is the parallel transport of** $Y(0)$ **along** x_t **with respect to** ∇^E. As before, $Y(1)$ is uniquely determined by $Y(0)$. We write

$$Y(1) = \tau^E(x,y)Y(0). \tag{3.2.82}$$

In this case, $\tau^E(x,y) : E_y \to E_x$ is a linear isomorphism.

Theorem 3.2.12. *There exists a unique formal solution $k_t(x,y)$ of the heat equation*

$$(\partial_t + H_x)k_t(x,y) = 0 \tag{3.2.83}$$

of the form

$$k_t(x,y) = q_t(x,y)j^{-1/2}(\mathbf{x})\sum_{i=0}^{\infty} t^i \Phi_i(x,y), \tag{3.2.84}$$

such that $\Phi_0(y,y) = \mathrm{Id}_E$. Furthermore, we have the following recursive formula for Φ_i:

$$\tau^E(x,y)^{-1}\Phi_i(x,y) = -\int_0^1 s^{i-1}\tau^E(x_s,y)^{-1}(B_x \cdot \Phi_{i-1})(x_s,y)ds. \tag{3.2.85}$$

In particular, $\Phi_0(x,y) = \tau^E(x,y)$.

Proof. From Definition 3.2.11, k_t in (3.2.84) is a formal solution if and only if

$$\left(\partial_t + B + t^{-1}\nabla^E_{\mathcal{R}}\right)\sum_{i=0}^{\infty} t^i \Phi_i(x,y) = 0. \tag{3.2.86}$$

Note that the equation (3.2.86) is equivalent to the system of equations:

$$\begin{aligned} &\nabla^E_{\mathcal{R}}\Phi_0 = 0, \\ &(\nabla^E_{\mathcal{R}} + i)\Phi_i = -B_x\Phi_{i-1}, \quad i > 0. \end{aligned} \tag{3.2.87}$$

The parallel transport $\tau^E(x, y)$ along x_s satisfies the equation $\nabla^E_{\mathcal{R}}\tau^E = 0$ and $\tau^E(y, y) = \mathrm{Id}_E$. So from the uniqueness of the differential equation with initial condition, we have $\Phi_0(x, y) = \tau^E(x, y)$.

Let $\{Y_{y,j}\}_j$ be a basis of E_y. Since $\tau^E(x_s, y) : E_y \to E_{x_s}$ is a linear isomorphism, $\{Y_{x_s,j} = \tau^E(x_s, y)Y_{y,j} \in E_{x_s}\}$ is a basis of E_{x_s}. We could write

$$\Phi_i(x_s) = X_j(s)Y_{x_s,j}, \quad -B_x\Phi_{i-1}(x_s) = Z_j(s)Y_{x_s,j}. \tag{3.2.88}$$

Since $\mathcal{R}(x_s) = s\dot{x}_s$ and $\nabla^E_{\mathcal{R}}Y_{x_s,j} = 0$, we have

$$\nabla^E_{\mathcal{R}}\Phi_i(x_s) = \mathcal{R}(X_j(s))Y_{x_s,j} = sX'_j(s)Y_{x_s,j}. \tag{3.2.89}$$

So the second equation in (3.2.87) is equivalent to the differential equation

$$sX'_j(s) + iX_j(s) = Z_j(s). \tag{3.2.90}$$

The solution of $X_j(s)$ in (3.2.90) is

$$X_j(s) = s^{-i}\int_0^s v^{i-1}Z_j(v)dv + Cs^{-i}. \tag{3.2.91}$$

Since we want to get good behavior for $s \to 0$, we take $C = 0$ in (3.2.91). Observe that $\tau^E(x, y)^{-1}\Phi_i(x, y) = X_j(1)Y_{y,j}$ and $\tau^E(x_s, y)^{-1}(B_x \cdot \Phi_{i-1})(x_s, y) = Z_j(s)Y_{y,j}$. We obtain (3.2.85).

The proof of Theorem 3.2.12 is completed. □

Let $\psi : \mathbb{R}_+ \to [0, 1]$ be a smooth cut-off function such that

$$\psi(s) = \begin{cases} 1, & \text{if } s < \varepsilon^2/4, \\ 0, & \text{if } s > \varepsilon^2. \end{cases} \tag{3.2.92}$$

We write $j(x, y) = j(\mathbf{x})$. Then we construct $k^N_t(x, y)$ in Theorem 3.2.3 by

$$k^N_t(x, y) = \psi(d(x, y)^2)q_t(x, y)j^{-1/2}(x, y)\sum_{i=1}^N t^i\Phi_i(x, y). \tag{3.2.93}$$

The following theorem is stronger than Theorem 3.2.3.

Theorem 3.2.13. *Let ℓ be an even positive integer.*

(1) For any $T > 0$, the kernels k_t^N, $0 < t < T$ define a uniformly bounded family of operators K_t^N on $\mathscr{C}^\ell(M, E)$, and

$$\lim_{t \to 0} \|K_t^N s - s\|_{\mathscr{C}^\ell} = 0. \tag{3.2.94}$$

(2) There exist differential operators D_k of order less than or equal to $2k$ such that D_0 is the identity and such that for any $s \in \mathscr{C}^{\ell+1}(M, E)$,

$$\left\| K_t^N s - \sum_{k=0}^{\ell/2-j} t^k D_k s \right\|_{\mathscr{C}^{2j}} = O\left(t^{(\ell+1)/2-j}\right). \tag{3.2.95}$$

(3) The kernel $r_t^N(x, y) = (\partial_t + H_x) k_t^N(x, y)$ satisfies the estimates

$$\|\partial_t^k r_t^N\|_{\mathscr{C}^\ell} < C t^{(N-n/2)-k-\ell/2}, \tag{3.2.96}$$

where the constant $C > 0$ only depends on ℓ and k.

Proof. We write $y = \exp_x \mathbf{y}$, with $\mathbf{y} \in T_xM$. Let

$$\Psi_i(x, \mathbf{y}) := \psi(\|\mathbf{y}\|^2) j^{1/2}(\mathbf{y}) \Phi_i(x, \exp_x \mathbf{y}) \tau^E(x, y)^{-1} \in \mathrm{End}(E_x). \tag{3.2.97}$$

Let $s \in \mathscr{C}^\infty(M, E)$. For $y \in B(x, \varepsilon)$, we write $s(x, \mathbf{y}) = \tau^E(x, y) s(y) \in E_x$. Then from (3.2.57), for $\mathbf{y} = \sqrt{t} v$,

$$\begin{aligned}(K_t^N s)(x) &= (4\pi t)^{-n/2} \int_M e^{-d(x,y)^2/4t} \sum_{i=1}^N t^i \psi(d(x,y)^2) j^{-1/2}(x,y) \Phi_i(x,y) s(y) dy \\ &= (4\pi t)^{-n/2} \int_{T_xM} e^{-|\mathbf{y}|^2/4t} \sum_{i=1}^N t^i \Psi_i(x, \mathbf{y}) s(x, \mathbf{y}) d\mathbf{y} \\ &= (4\pi)^{-n/2} \int_{T_xM} e^{-|v|^2/4} \sum_{i=1}^N t^i \Psi_i(x, t^{1/2} v) s(x, t^{1/2} v) dv. \end{aligned} \tag{3.2.98}$$

From (3.2.97), we see that $\Psi_0(x, 0) = \mathrm{Id}_{E_x}$.

Let $\{Y_j\}$ be a basis of E_y and $s(y) = s_j Y_j$. Then $s(x, \mathbf{y}) = s_j \tau^E(x, y) Y_j$. Let $C = \max_{y \in B(x,\varepsilon)} \sum_{|\alpha| \le \ell} |D_x^\alpha(\tau^E(x, y))|$. Then $\|s(x, \mathbf{y})\|_{\mathscr{C}_x^\ell} \le C|s(y)| \le C\|s\|_{\mathscr{C}^0} \le C\|s\|_{\mathscr{C}^\ell}$. So from (3.2.98), there exists $C' > 0$ such that for $0 \le t \le T$,

$$\|K_t^N s\|_{\mathscr{C}^\ell} \le C' \|\Psi\|_{\mathscr{C}_x^\ell} \left(\sum_{i=1}^N T^i \right) \left(\int_{\mathbb{R}^n} e^{-|v|^2/4} dv \right) \cdot \|s\|_{\mathscr{C}^\ell}. \tag{3.2.99}$$

So K_t is uniformly bounded on $\mathscr{C}^\ell(M,E)$. In this case, for $|\alpha| \le \ell$,

$$\lim_{t\to 0} D^\alpha(K_t^N s - s) = D^\alpha \lim_{t\to 0}(K_t^N s - s). \tag{3.2.100}$$

From (3.2.98),

$$\lim_{t\to 0} K_t^N s = (4\pi)^{-n/2}\int_{T_xM} e^{-|v|^2/4}\Psi_0(x,0)s(x,0)dv = s(x). \tag{3.2.101}$$

Therefore, we get (3.2.94).

For (2), set $\sigma = t^{1/2}$ and

$$f(\sigma, v) = \sum_{i=1}^{N} \sigma^{2i}\Psi_i(x,\sigma v)s(x,\sigma v). \tag{3.2.102}$$

Taylor expansion at $\sigma = 0$, from (1.3.43), we have

$$\begin{aligned} f(\sigma,v) = \sum_{|\alpha|\le l}\sum_{\beta+\gamma=\alpha}\sum_{i=1}^{N}\sigma^{2i}\frac{\alpha!}{\beta!\gamma!}\Psi_i^{(\beta)}(x,0)s^{(\gamma)}(x,0)(\sigma v)^\alpha \\ + \sum_{|\mu|=l+1}\sum_{\beta+\gamma=\mu}\sum_{i=1}^{N}\sigma^{2i}\frac{l+1}{\mu!}(\sigma v)^\mu\cdot\frac{\alpha!}{\beta!\gamma!} \\ \cdot\int_0^1(1-s)^l\Psi_i^{(\beta)}(x,sv)s^{(\gamma)}(x,sv)ds. \end{aligned} \tag{3.2.103}$$

If $|\alpha| \le \ell$ is odd,

$$\int_{T_xM} e^{-|v|^2/4}v^\alpha dv = 0. \tag{3.2.104}$$

So we only need to consider the even case. In this case, let

$$D_k := (4\pi)^{-n/2}\sum_{2i+|\alpha|=2k}\sum_{|\alpha|\le l}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\Psi_i^{(\beta)}(x,0)\int_{T_xM} e^{-|v|^2/4}v^\alpha dv. \tag{3.2.105}$$

In particular, $D_0 = \mathrm{Id}$. Since $s^{(\gamma)}(x,0) = D_x^\gamma s(x)$, from (3.2.98), (3.2.103) and (3.2.105), we have

$$\left\| K_t^N s - \sum_{k=0}^{\ell/2} t^k D_k s\right\|_{\mathscr{C}^0} = O\left(t^{(\ell+1)/2}\right). \tag{3.2.106}$$

In the same way, we have

$$\left\| K_t^N s - \sum_{k=0}^{\ell/2-j} t^k D_k s \right\|_{\mathscr{C}^{2j}} = O\left(t^{(\ell+1)/2-j}\right). \tag{3.2.107}$$

Note that in this case, in the reminder term of (3.2.6), we need to estimate $\|s\|_{\mathscr{C}^\ell}$. So our estimate is for $O\left(t^{(\ell+1)/2-j}\right)$.

From Propositions 3.2.9 and 3.2.10,

$$r_t^N(x,y) = q_t(x,y) j^{-1/2}(x,y)\Big(\partial_t + B_x + t^{-1}\nabla^E_{\mathcal{R}}\Big)\left(\psi(d(x,y)^2)\sum_{i=1}^{N} t^i \Phi_i(x,y)\right). \tag{3.2.108}$$

If $d(x,y) \le \varepsilon/2$, $\partial_x^\alpha \psi(d(x,y)^2) \equiv 0$. If $d(x,y) > \varepsilon/2$, for any k, we have

$$\partial_x^\alpha \psi(d(x,y)^2) e^{-d(x,y)^2/4t} < \partial_x^\alpha \psi(d(x,y)^2) e^{-\varepsilon^2/16t} = O(t^k). \tag{3.2.109}$$

From (3.2.86), the terms on the right hand side of (3.2.108), which do not involve a derivative of $\psi(d(x,y)^2)$ cancel, except for one remaining term $t^N q_t(x,y)(B_x\Phi_N)(x,y)$, which may be bounded by $t^{N-n/2}$. So we have

$$\|r_t^N\|_{\mathscr{C}^0} < Ct^{(N-n/2)}. \tag{3.2.110}$$

The estimate of $\|\partial_t^k r_t^N\|_{\mathscr{C}^\ell}$ is similar, once we observe that

$$\partial_t e^{-x^2/t} = t^{-1}(x^2/t)e^{-x^2/t} = O(t^{-1}), \tag{3.2.111}$$

and

$$\partial_x e^{-x^2/t} = t^{-1/2}(-2x/t^{1/2})e^{-x^2/t} = O(t^{-1}). \tag{3.2.112}$$

The proof of Theorem 3.2.13 is completed.

□

Now we summarize the properties of heat kernels.

Theorem 3.2.14. *Let $p_t(x,y)$ be the heat kernel of H. Then there exist $\Phi_i \in \mathscr{C}^\infty(M\times M, E\boxtimes E^*)$ such that for every $N > n/2$, the kernel $k_t^N(x,y)$ defined by*

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}} \psi(d(x,y)^2) j^{-1/2}(x,y) \sum_{i=1}^{N} t^i \Phi_i(x,y) \tag{3.2.113}$$

is asymptotic to $p_t(x,y)$:

$$\|\partial_t^k (p_t(x,y) - k_t^N(x,y))\|_{\mathscr{C}^\ell} = O(t^{N-n/2-\ell/2-k}). \tag{3.2.114}$$

The leading term $\Phi_0(x,y) = \tau^E(x,y)$.

The following corollary is the generalization of Proposition 3.1.6.

Corollary 3.2.15. *Let P_t be the heat operator associated with $p_t(x, y)$. Then for $k \in \mathbb{N}$, $s \in \mathscr{C}^\infty(M, E)$,*

$$\left\| P_t s - \sum_{i=0}^{k} \frac{(-tH)^i}{i!} s \right\|_{\mathscr{C}^j} = O(t^{k+1}). \tag{3.2.115}$$

Proof. The heat equation $(\partial_t + H)P_t s = 0$ implies that in (3.2.95), $D_k = (-H)/k!$.

The proof of Corollary 3.2.15 is completed. □